SOLUTION OF AN INVERSE PROBLEM OF NONLINEAR HEAT CONDUCTION TO DETERMINE THERMOPHYSICAL CHARACTERISTICS

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A problem to determine the temperature dependence of the coefficient of heat conduction is formulated on the basis of an analysis of the internal temperature field. A search method permitting solution of the problem mentioned is proposed.

There is a large quantity of work devoted to the investigation of methodological questions associated with the determination of the thermophysical characteristics of materials at high temperatures at this time. As a rule, methods based on using exact and approximate solutions of a nonlinear heat-conduction equation for particular cases are examined in these papers. The practical utilization of the mentioned methods is related to the realization of stationary heating of the specimen of material under investigation, monotonic heating [1], instantaneous or intensive heating to a given temperature [2, 3], which is quite tedious and sometimes even unrealizable under laboratory conditions in practice. In this connection, methods to determine the thermophysical characteristics [4-6], based on an analysis of the temperature fields within the material under investigation by numerical solution of the inverse problem of nonlinear heat conduction, which require no special conditions for conducting the experiment, become quite valuable. A method is elucidated below for the determination of the coefficient of heat conduction as a function of the temperature, on the basis of a numerical solution of the inverse problem of heat conduction using a refined difference scheme and a special direct search method. An example of the use of the proposed method on the basis of a numerical experiment is presented.

The problem of determining the thermophysical characteristics by means of experimentally measured temperature fields can be formulated as follows: The heat-propagation process is described by

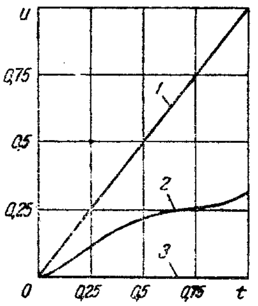

Fig. 1

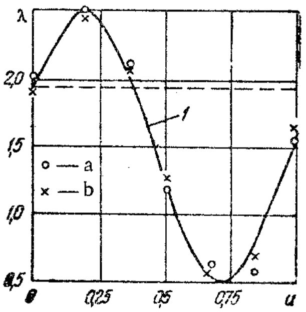

Fig. 2

Fig. 1. Time change in the temperature: 1) u(o, t); 2) u(1/2, t); 3) u(1, t).

Fig. 2. Comparison with the exact solution. 1) Exact solution; a) obtained solution, linear interpolation; b) obtained solution, quadratic interpolation; the dashes denote the first approximation.

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TABLE 1. Comparison with the Exact Solution

x		0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\bar{u}_r$		1	0,9512	0,9048	0,8607	0,8187	0,7788	0,7408	0,7047	0,6703	0,6376	0,6056
$\alpha_k^{j-1} \neq 1$	u	1	0,9464	0,8964	0,8498	0,8066	0,7664	0,7291	0,6941	0,6625	0,6329	0,6056
$\beta_k^{j-1} \neq 1$	Δu	0	0,0048	0,0084	0,0109	0,0121	0,0124	0,0117	0,0102	0,0078	0,0047	0
$\alpha_k^{j-1} \equiv 1$	u	1	0,9291	0,8662	0,8110	0,7680	0,7218	0,6870	0,6581	0,6351	0,6176	0,6056
$\beta_k^{j-1} \equiv 1$	Δu	0	0,0221	0,0386	0,0497	0,0507	0,057	0,0538	0,0466	0,0352	0,020	0

the nonlinear equation of heat conduction

$$\frac{\partial}{\partial x}\lambda(u)\frac{\partial u}{\partial x} = c\gamma(u)\frac{\partial u}{\partial t}, \quad 0 < x < \delta, \quad 0 < t \leqslant t_k. \tag{1}$$

The time change in the temperature $f_i(t)$ is known (as a result of measurements during the experiment) at several internal points x_i, $i = 1, 2, \ldots, n$; $0 \leq x_i \leq \delta$. It is required to determine the unknown function $\lambda(u)$ from the condition of the minimum of the functional

$$J = \int_0^T \sum_1^n [f_i(t) - u(x_i, t)]^2 dt \tag{2}$$

or

$$J = \int_0^T \sum_1^n \left[\frac{f_i(t) - u(x_i t)}{f_i(t)}\right]^2 dt. \tag{2'}$$

where $u(x_i, t)$ is the temperature change at the point x_i computed by means of (1). The initial and boundary conditions needed to solve (1) should be given. The temperature changes measured at the extreme points or the true conditions of the experiment can be used as boundary conditions if $n \geq 3$. The procedure for solving the problem is to assign the form of the function $\lambda(u)$ a priori, to determine those variations $\delta\lambda(u)$ which diminish the value of the functional (2), and to vary $\lambda(u)$ in this direction until min J is obtained. The values of the functional are determined by numerical solution of the direct problem, i.e., Eq. (1) for a selected $\lambda(u)$. An implicit conservative difference scheme recommended in [7] was used to solve the direct problem. However, the supplementary coefficients α_k^{j-1}, β_k^{j-1}

$$\frac{\partial u}{\partial x} = \alpha_k^{j-1}\frac{u_k^j - u_{k-1}^j}{\Delta x_k}, \quad \frac{\partial^2 u}{\partial x^2} = \beta_k^{j-1}\frac{u_{k+1}^j - 2u_k^j + u_{k-1}^j}{\Delta x_k^2}$$

were introduced to increase the accuracy in the approximating difference expressions for the derivatives $\partial u/\partial x$ and $\partial^2 u/\partial x^2$: The subscript k refers to the partition in x, and j refers to t. The coefficients α_k^{j-1} and β_k^{j-1} are determined by means of the temperature values at points of the mesh domain and are the ratio between the first- and second-order derivatives determined at five points by means of the Lagrange formulas [8] and the corresponding derivatives determined at two points for the first, and at three points for the second, derivatives. An analysis of the computations made showed that introduction of the coefficients mentioned is most effective for the case of large temperature gradients. Given in Table 1 is a comparison between the results of numerical computations using an ordinary difference scheme ($\alpha_k^{j-1} \equiv 1$, $\beta_k^{j-1} \equiv 1$) and using the proposed scheme ($\alpha_k^{j-1} \not\equiv 1$, $\beta_k^{j-1} \not\equiv 1$) with the exact quasistationary solution $\bar{u}_r$ of the problem with a moving boundary [9].

A specially developed search method is used to solve the problem of minimizing the functional (2). The crux of the method is the following. The whole range of temperature variation is partitioned into m parts by the points $u_{min} = u_1 < u_2 < \ldots < u_m = u_{max}$. The values of the desired function $\lambda(u_i) = \lambda_i$ are to be determined. It is assumed that the law of variation of λ between the nodal points is given. Linear interpolation between two adjacent nodes and parabolic interpolation between three nodes are used. Taken as the first approximation to the solution of the initial problem is $\lambda(u) = \lambda^{(1)} = \text{const}$. The search is accomplished as follows. Three constant values $\bar{\lambda}_1 < \bar{\lambda}_2 < \bar{\lambda}_3$ are selected such that their corresponding values of the quality criterion

(2) would satisfy the inequality

$$J(\bar{\lambda}_1) < J(\bar{\lambda}_2) < J(\bar{\lambda}_3).$$

Assuming the dependence $J(\lambda)$ in the neighborhood can be represented by a parabola, we find $\bar{\lambda}_1 < \bar{\lambda} < \bar{\lambda}_3$ which yields $\lambda_0^{(1)}$ in the neighborhood mentioned. Then, an analogous procedure is repeated in a smaller neighborhood of the point $\min J(\lambda)$ and so on until $\lambda_0^{(1)}$ is obtained with a given accuracy. The successive approximations are determined as follows. Let the (k − 1)-th approximation, i.e., $\lambda_1^{(k-1)}$, $\lambda_2^{(k-1)}$, ..., $\lambda_m^{(k-1)}$, be known. To determine the k-th approximation, a certain increment $d\lambda = \lambda^{(1)}/q$ (q is the scale) is selected and the next increments of all the $\lambda_i^{(k-1)}$, starting with i = 0, are given successively:

$$1.\ \lambda_i^{(k)} = \lambda_i^{(k-1)} \pm d\lambda, \quad 2.\ \lambda_i^{(k)} = \lambda_i^{(k-1)} \pm d\lambda/2, \quad 3.\ \lambda_i^{(k)} = \lambda_i^{(k-1)}.$$

The value which yields the minimal value of the quality criterion (2)

$$J(\lambda_1^{(k)}, \ldots, \lambda_i^{(k)}, \lambda_{i+1}^{(k-1)}, \ldots, \lambda_m^{(k-1)})$$

is selected as $\lambda_i^{(k)}$. The mentioned procedure is repeated until it turns out that $\lambda_i^{(k)} = \lambda_i^{(k-1)}$ after which the scale q is increased and the next approximations are determined. The search is considered terminated if the scale becomes greater than a given maximum quantity.

Presented below as an illustration of the proposed method is an example based on a numerical experiment. Shown in dimensionless form in Fig. 1 is the time change in the temperature at the points x = 0, x = 1/2, x = 1. The temperature change at the point x = 1/2 has been obtained by numerical solution of (1) for the case

$$u(x,\ 0) = 0; \quad u(0,\ t) = t; \quad u(1,\ t) = 0; \quad c\gamma = 1;$$
$$\lambda(u) = \sin(6u + 0.4) + 1.5.$$

The results of solving the problem when using the criterion in the form (2') are shown in Fig. 2.

NOTATION

c, specific heat; f, temperature at an internal point; J, functional; m, number of partitions in u; n, number of temperature measurements; q, scale; t, time; u, temperature; $\bar{u}_T$, exact solution; x, linear coordinate; α, correction coefficient for the first derivative; β, correction coefficient for the second derivative; γ, specific gravity; δ, thickness; λ, coefficient of heat conduction; $\bar{\lambda}$, constant value of λ. Subscripts: i refers to partition in u; j refers to partition in t; k refers to partition in x; (k) refers to an approximation.

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